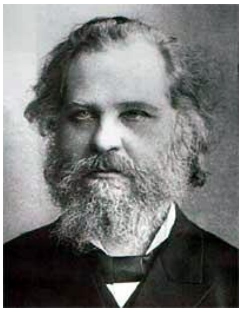
Umov's theorem and Umov's vector



If we consider surrounding (non-isolated thermodynamic system), that on its border is possible heat exchange by every three methods and in consequence this exchange there are occurred changing of particular components of surrounding energy, corresponding non-steady process is expressed by differential equation in form

$$\frac{\partial}{\partial t}\left(w_u + w_\mathrm{k} + w_\mathrm{p} + w_\mathrm{r}\right) + \mathrm{div}\left(\boldsymbol{q}_\mathrm{v} + \boldsymbol{q}_\mathrm{k} + \boldsymbol{q}_\mathrm{r}\right) = q_\mathrm{z} \qquad [\mathrm{W.m^{-3}}] \qquad (1)$$

First term of equation (1) represents time change of particular energy forms in surrounding, that are expressed by their volume densities, namely:

- internal energy density $w_u = r \cdot c \cdot J$ [J.m^{-3}] (2)
- kinetic energy density $w_\mathrm{k} = \frac{1}{2} \cdot r \cdot v^2$ (3)
- potential energy density $w_\mathrm{p} = \sum_{i=1}^{n} r_i \cdot w_{\mathrm{p}i}$ (4)
- radiant energy density w_r [J.m^{-3}] (5)

The second term represents energy fluxes into or from surrounding by conduction, convection and radiation, expressed by their densities.

And the third term q_z [W.m^{-3}] is power density of internal source, if it exists in surrounding.

Equation (1) represents energetic balance of surrounding that is in interaction with surrounding (law of conservation of energy in thermodynamic non-insulated system). This equation is also known as a ***general differential equation of energy propagation*** in various surroundings.

Differential equation for energy transfer by radiation

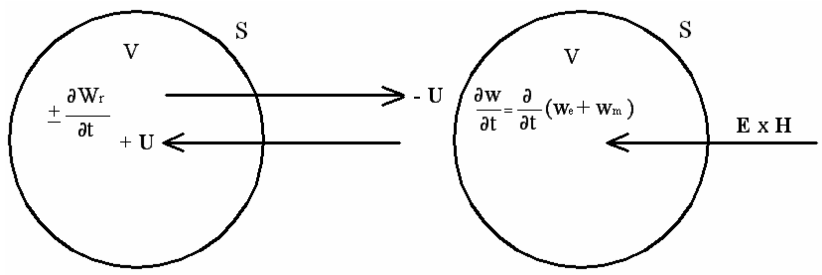
$$\frac{\partial w_\mathrm{r}}{\partial t} + \mathrm{div}\,\boldsymbol{q}_\mathrm{r} = 0 \qquad (5)$$

what is equation of radiant energy propagation in thermodynamic equilibrium.
After specifying divergence operator according to space coordinates we get equation

$$\frac{\partial w_\mathrm{r}}{\partial t} + \frac{\partial q_{\mathrm{r},x}}{\partial x} + \frac{\partial q_{\mathrm{r},y}}{\partial y} + \frac{\partial q_{\mathrm{r},z}}{\partial z} = 0 \qquad (6)$$

with this physical meaning: In radiant surrounding there always exist three functions q_x, q_y, q_z, with the property, that sum of their first derivation according to coordinates *x*, *y*, *z* determines decreasing of energy density at particular surrounding point per time unit. These functions represent energy flow densities. This theorem was presented by Russian physicist Nikolay Alekseevich Umov in 1874 and it expresses law of radiant (electromagnetic) energy conservation. Freely, it can be interpreted also in this form (figure below): By flowing (inflow or outflow) of radiant energy through the closed surface with area *S* of particular surrounding with volume *V*, there are occurred changes of particles energy of this surrounding (increasing or decreasing). If we express again the surrounding energy W_r by vector of its volume density the mathematical formulation of Umov's theorem will be also:

$$\oint_S \boldsymbol{q}_\mathrm{r}\, \mathrm{d}S = -\frac{\partial W_\mathrm{r}}{\partial t} = -\frac{\partial}{\partial t}\int w_\mathrm{r} \cdot \mathrm{d}V \tag{7}$$



Umov's vector *Poynting's vector*

Because right hand side of equation (7) physically means the energy change of surrounding per time unit, to objective equation belongs also statement, that the energy flow through the surrounding is equals to velocity of energy change in this surrounding v_w

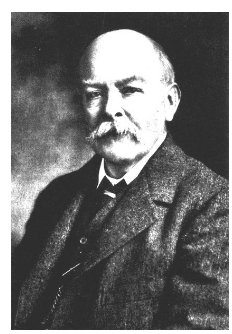
$$\oint_S \boldsymbol{q}_\mathrm{r} \cdot \mathrm{d}S = -\int_V v_w \cdot \mathrm{d}V \tag{8}$$

In equations (7), eventually (8) the vector of radiant energy flow density $\boldsymbol{q}_\mathrm{r}$ is used to name ***Umov's vector*** and is denoted $\boldsymbol{U}$. In differential form the Umov's theorem is

$$\mathrm{div}\, \boldsymbol{q}_\mathrm{r} \equiv \mathrm{div} \boldsymbol{U} = -\frac{\partial w_\mathrm{r}}{\partial t} = -v_w \tag{9}$$

or simply $U = w_\mathrm{r} \cdot v$, what also means, that Umov's vector is equals to product of volume density of radiant energy and its propagation velocity v [m.s^{-1}]. Umov's vector is always dependant on surrounding properties and on energy propagation nature.

Poynting's radiant vector



In theory of electric heating is usable Umov's vector as an instrument for expression of transfer of electromagnetic field energy and its conversion to heat in heated material. If we expose any material, conductive or non-conductive, magnetic or non-magnetic, in solid or in other phase, to effect of electromagnetic field, the energy flow to material is influenced by quantities, which explicitly define field. Such quantities are electric and magnetic component of electromagnetic field intensity.

Volume energy densities of both components of electromagnetic field are:

- in steady electric field

$$w_\mathrm{e} = \frac{1}{2} \cdot \boldsymbol{D} \cdot \boldsymbol{E} = \frac{1}{2} \cdot e \cdot \boldsymbol{E}^2 \tag{10}$$

- v steady magnetic field

$$w_\mathrm{m} = \frac{1}{2} \cdot \boldsymbol{B} \cdot \boldsymbol{H} = \frac{1}{2} \cdot m \cdot \boldsymbol{H}^2 \tag{11}$$

Sum of (10) and (11) determines total volume energy density of electromagnetic field

$$w = \frac{1}{2} \cdot \left(\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H}\right) = \frac{1}{2} \cdot \left(e \cdot \boldsymbol{E}^2 + m \cdot \boldsymbol{H}^2\right) \qquad [\mathrm{J.m^{-3}}] \tag{12}$$

If we apply this equation to non-steady field, then the corresponding time derivatives of both components of field energy physically represent their change, otherwise the transfer of volume density of field energy per time unit

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(w_{\mathrm{e}} + w_{\mathrm{m}}\right) = e \cdot \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + m \cdot \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t} \qquad [\mathrm{W.m^{-3}}] \qquad (13)$$

Let's equation (13) specify for non-conductive surrounding (g = 0):

Using of Maxwell's equations the equation (13) is transformed to either differential form

$$\frac{\partial w}{\partial t} = \boldsymbol{E} \cdot \mathrm{rot}\, \boldsymbol{H} - \boldsymbol{H} \cdot \mathrm{rot}\, \boldsymbol{E} = -\mathrm{div}\left(\boldsymbol{E} \times \boldsymbol{H}\right) \qquad (14)$$

or to integral form (for whole volume of non-conductive surrounding)

$$\frac{\partial}{\partial t}\int_V w \cdot \mathrm{d}V + \oint_S \left(\boldsymbol{E} \times \boldsymbol{H}\right) \cdot \mathrm{d}\boldsymbol{S} = 0 \qquad (15)$$

Physical interpretation of the last equation is: energy increasing in volume of non-conductive surrounding per time unit is equals to energy flow into this volume through the envelope area *S*, transported in electromagnetic form. It is equation of energetic balance of non-conductive surrounding, expressed by powers. Energy flow density is characterized by vector product $\boldsymbol{E} \times \boldsymbol{H}$, which was introduced to electromagnetic field theory by English physicist John Henry Poynting in 1885. It is named ***Poynting's radiant vector***. If we compare equation (7) with equation (15), eventually (9) with (14) we very soon discover, that they are the same. It means, that the Poynting's radiant vector is Umov's vector, expressed by ***E*** and ***H***, components of electromagnetic field intensities. Therefore for non-conductive surrounding is

$$\boldsymbol{N} \equiv \boldsymbol{U} = \boldsymbol{S}_N = \boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{E} \times \frac{\boldsymbol{B}}{m_0} = \frac{1}{m_0} \cdot \boldsymbol{E} \times \boldsymbol{B} \quad [\mathrm{W.m^{-2}}] \qquad (16)$$

where vector $\boldsymbol{S}_N$ is in direction of wave propagation. As fields ***E*** and ***B*** are on each other perpendicular and in non-conductive surrounding are spread by speed of light *c*, we can very easy check, that size of vector $\boldsymbol{S}_N$ is (by applying expression $E = c \cdot B$ and $c = \frac{1}{\sqrt{m_0 \cdot e_0}}$)

$$\left|\boldsymbol{S}_N\right| = \frac{\left|\boldsymbol{E} \times \boldsymbol{B}\right|}{m_0} = \frac{E \cdot B}{m_0} = c \cdot \left(\frac{1}{2} \cdot e_0 \cdot E^2 + \frac{1}{2} \cdot \frac{B^2}{m_0}\right) = \frac{c \cdot B^2}{m_0} = c \cdot e_0 \cdot E^2 = S_N \qquad (17)$$

By that fact, that we deduce Poynting's radiant vector for non-conductive surrounding, it means, that it is implementing only with this condition. On the other side, in respect to electro-heat conversion it has essential sense also for conductive surrounding, as a suitable instrument for expression of electromagnetic field energy transfer to such surrounding and by that fact also for heat generation in it.

Example 1

At upper layer of Earth atmosphere is average value of Poynting's vector $\overline{S}_N$ = 1,35.10^3 W.m^{-2}. This value is named *solar constant*.

a) What are the magnitudes of electric and magnetic field in condition, that sun electromagnetic radiation is plane sine wave?
b) What is total average power radiated from the Sun? Average distance Earth-Sun is r_0 = 1,5.10^{11} m.

Solution:

a) Average value of Poynting's vector is related to magnitude of electric field according to expression

$$\overline{S}_N = c \cdot \frac{1}{2} \cdot e_0 \cdot E_{\max}^2$$

Hence, expression for magnitude of electric field

$$E_{\max} = \sqrt{\frac{2 \cdot \overline{S}_N}{c \cdot e_0}} = 1{,}01 \cdot 10^3 \text{ V.m}^{-1}$$

Corresponding magnitude of magnetic field is

$$B_{\max} = \frac{E_{\max}}{c} = 3{,}4 \cdot 10^{-6} \text{ T}$$

Note: this magnetic field is smaller than 1/10 of magnetic field of the Earth.

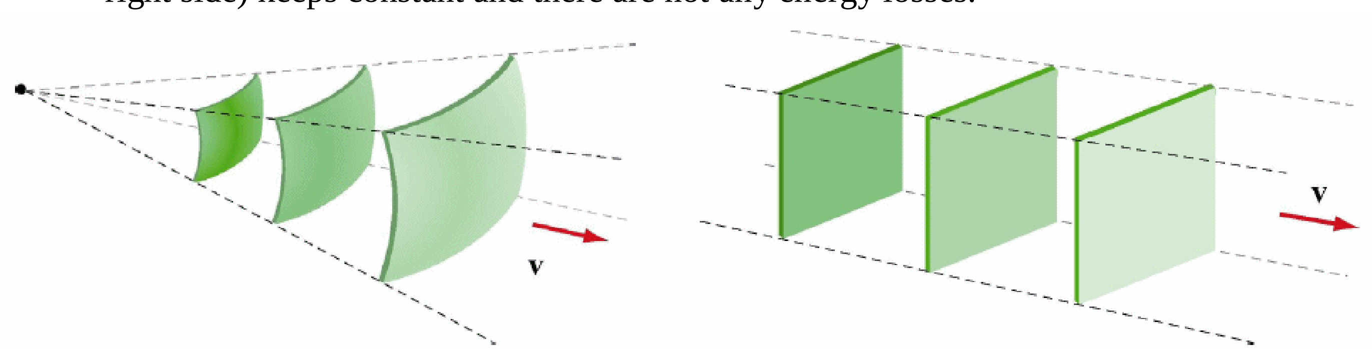
b) Total average power radiated from the Sun in distance r_0 is

$$\overline{P} = \overline{S}_N \cdot S = \overline{S}_N \cdot \left(4 \cdot p \cdot r_0^2\right) = 3{,}8 \cdot 10^{26} \text{ W}$$

Wave type, in this case, is sphere wave (Fig. on the left side). This wave rise from the "point" source. Intensity at distance *r* from the source is

$$I = \overline{S}_N = \frac{\overline{P}}{4 \cdot p \cdot r^2}$$

and it decreases as the function 1/r^2. On the other side, intensity of plane wave (Fig. on the right side) keeps constant and there are not any energy losses.



Example 2

Determine the value of Poynting's vector of standing electromagnetic wave that is given by expression

$$E_y(x,t) = 2 \cdot E_{\max} \cdot \cos(k \cdot x) \cdot \cos(w \cdot t) \qquad B_z(x,t) = 2 \cdot B_{\max} \cdot \sin(k \cdot x) \cdot \sin(w \cdot t)$$

Solution:

Poynting's vector of standing wave is

$$\boldsymbol{S}_N = \frac{\boldsymbol{E} \times \boldsymbol{B}}{m_0} = \frac{1}{m_0} \cdot \left[2 \cdot E_{\max} \cdot \cos(k \cdot x) \cdot \cos(w \cdot t) \cdot \boldsymbol{j}\right] \times \left[2 \cdot B_{\max} \cdot \sin(k \cdot x) \cdot \sin(w \cdot t) \cdot \boldsymbol{k}\right] =$$

$$= \frac{4 \cdot E_{\max} \cdot B_{\max}}{m_0} \cdot [\sin(k \cdot x) \cdot \cos(k \cdot x) \cdot \sin(w \cdot t) \cdot \cos(w \cdot t)] \cdot \boldsymbol{i} =$$

$$= \frac{E_{\max} \cdot B_{\max}}{m_0} \cdot [\sin(2 \cdot k \cdot x) \cdot \sin(2 \cdot w \cdot t)] \cdot \boldsymbol{i}$$

Time average value $\overline{S}_N$ is

$$\overline{S}_N = \frac{E_{\max} \cdot B_{\max}}{m_0} \cdot \sin(2 \cdot k \cdot x) \cdot \sin(2 \cdot w \cdot t) = 0$$

This result was expected, because standing wave is not propagated. Therefore, energies, that are transferred by two waves propagated in opposite direction so, that they create standing waves, and they are together interrupted. It means that there is no energy propagation.

Example 3

Consider, that electric field of plane electromagnetic wave is given by expression

$$E(z,t) = E_{\max} \cdot \cos(k \cdot z - w \cdot t) \cdot \boldsymbol{i}$$

Determine following values:

a) Direction of wave spreading ($\boldsymbol{S}_N$).
b) Equivalent magnetic field $\boldsymbol{B}$.

Solution:

a) After rewriting argument of cosine function $(k \cdot z - w \cdot t) = k \cdot (z - c \cdot t)$, where $w = c \cdot k$, and we can see from there, that direction of wave propagation is $+z$.

b) Direction of electromagnetic wave propagation is the same as direction of Poynting's vector, which is given by $\boldsymbol{S}_N = \frac{\boldsymbol{E} \times \boldsymbol{B}}{m_0}$. Moreover we know, that fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are together perpendicular. Therefore $\boldsymbol{E} = E(z,t) \cdot \boldsymbol{i}$ and $\boldsymbol{S}_N = S_N \cdot \boldsymbol{k}$. Then $\boldsymbol{B} = B(z,t) \cdot \boldsymbol{j}$. It means, that $\boldsymbol{B}$ is propagated in positive direction of coordinates $+y$. Because $\boldsymbol{E}$ and $\boldsymbol{B}$ are on each other in the same phase, we can then write

$$\boldsymbol{B}(z,t) = B_{\max} \cdot \cos(k \cdot z - w \cdot t) \cdot \boldsymbol{j}$$

For finding magnitude of $\boldsymbol{B}$ we use Faraday's law:

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

which leads to

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}.$$

From these equations we get

$$-E_{\max} \cdot k \cdot \sin(k \cdot z - w \cdot t) = -B_{\max} \cdot w \cdot \sin(k \cdot z - w \cdot t)$$

or

$$\frac{E_{\max}}{B_{\max}} = \frac{w}{k} = c.$$

Magnetic field is then given by expression

$$\boldsymbol{B}(z,t) = \left(\frac{E_{\max}}{c}\right) \cdot \cos(k \cdot z - w \cdot t) \cdot \boldsymbol{j}$$