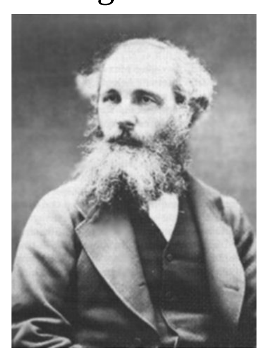
Wave equations of electromagnetic field

The electromagnetic field is a physical field that is produced by electrically charged objects and which affects the behavior of charged objects in the vicinity of the field. The electromagnetic field extends indefinitely throughout space and describes the *electromagnetic interaction*, one of the four fundamental forces of nature. The field can be viewed as the combination of an ***electric field*** and a ***magnetic field***. *The electric field* is produced by stationary charges, and *the magnetic field* by moving charges (currents); these two are often described as the sources of the field. Electric component is characterized by ***electric field intensity E***, magnetic component is normally expressed by ***magnetic field intensity H***. Vectors of these fields are on each other perpendicular.

Note: magnetic field is usually denoted by the symbol ***B***. Historically; ***B*** was called the ***magnetic flux density*** or ***magnetic induction***. A distinct quantity, ***H***, was called the magnetic field (strength), and this terminology is still often used to distinguish the two in the context of magnetic materials (non-trivial permeability *m*). Otherwise, however, this distinction is often ignored, and *both* quantities are frequently referred to as "the magnetic field." (Some authors call the *auxiliary field*, instead.)

Maxwell's equations – are a set of four partial differential equations of electromagnetic theory that express (1) how electric currents and changing electric fields produce magnetic fields (Ampère's Circuital Law), (2) how changing magnetic fields produce electric fields (Faraday's law of induction), (3) how electric charges produce electric fields (Gauss's law), and (4) the experimental absence of magnetic monopoles. For fixed surrounding – space filled with material that is homogenous, isotropic and linear, there are these equations stated in following order and form:



$$\mathrm{rot}\ \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = g \cdot \boldsymbol{E} + e \cdot \frac{\partial \boldsymbol{E}}{\partial t} \qquad (1)$$

$$\mathrm{rot}\ \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -m \cdot \frac{\partial \boldsymbol{H}}{\partial t} \qquad (2)$$

$$\mathrm{div}\ \boldsymbol{D} = r_0 \qquad (3)$$

$$\mathrm{div}\ \boldsymbol{B} = 0 \qquad (4)$$

These four Maxwell's equations are often completed with material’s equations that express surrounding influence to effects in electromagnetic field. For fixed surrounding (see above), they are defined as:

$$\boldsymbol{J} = g \cdot \boldsymbol{E} \qquad \boldsymbol{D} = e \cdot \boldsymbol{E} \qquad \boldsymbol{B} = m \cdot \boldsymbol{H} \qquad (5)$$

where g is electric conductivity [$S.m^{-1}$]
e is permitivity [$F.m^{-1}$]
m Is permeability [$H.m^{-1}$]

If in electromagnetic field is any free electric charge, then $r_0 = 0$ and also

$$\mathrm{div}\ \boldsymbol{D} = 0 \qquad (6)$$

In respect to modeling of electric heating it is useful substitute Maxwell’s equations (1) and (2) with that type of equations, in which will be only one component of electromagnetic intensity, i.e. either electric ***E*** or magnetic ***H***. This type of equations is named ***general equations***

of electromagnetic wave propagation (in next EMW) or ***wave equations of electromagnetic field*** (in next EMF). Together with unicity conditions they create mathematical model of EMF propagated in space, therefore model of EMW. Transformation of Maxwell's equations to wave equations is simple; procedure for magnetic component of EMF is indicated below:
At first; on 1. Maxwell's equation (1) we apply another rotation:

$$\text{rot}(\text{rot}\ \boldsymbol{H}) = \text{rot}(g \cdot \boldsymbol{E}) + \text{rot}\left(e \cdot \frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{7}$$

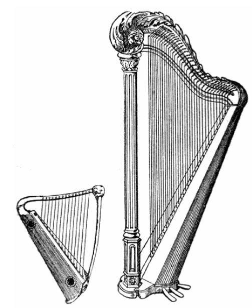
Introducing one simplification, that material quantities are constants, we can take them in front of expression:

$$\text{rot}(\text{rot}\ \boldsymbol{H}) = g \cdot \text{rot}(\boldsymbol{E}) + e \cdot \text{rot}\left(\frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{8}$$

Substituting of 2. Maxwell's equation (2) to expression (8) we get:

$$\text{rot}(\text{rot}\ \boldsymbol{H}) = -g \cdot m \cdot \frac{\partial \boldsymbol{H}}{\partial t} - e \cdot m \cdot \frac{\partial^2 \boldsymbol{H}}{\partial t^2} \tag{9}$$

In next computation will be utilized mathematic operator nabla ∇, which for vector operations is in form:



$$\nabla = \frac{\partial}{\partial x} \cdot \boldsymbol{i} + \frac{\partial}{\partial x} \cdot \boldsymbol{j} + \frac{\partial}{\partial x} \cdot \boldsymbol{k}$$

for ***gradient*** of scalar function f:

$$\text{grad f} = \nabla \text{f} = \frac{\partial \text{f}}{\partial x} \cdot \boldsymbol{i} + \frac{\partial \text{f}}{\partial x} \cdot \boldsymbol{j} + \frac{\partial \text{f}}{\partial x} \cdot \boldsymbol{k}$$

$\nabla(\text{f} \cdot \text{g}) = \text{f} \cdot (\nabla \text{g}) + \text{g} \cdot (\nabla \text{f})$ (formulation similar to 1. derivation of product)

for ***divergence*** of vector function $\boldsymbol{v}(x, y, z) = v_x \cdot \boldsymbol{i} + v_y \cdot \boldsymbol{j} + v_z \cdot \boldsymbol{k}$:

$$\text{div}\ \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

for ***rotation*** (curl) of vector function $\boldsymbol{v}(x, y, z) = v_x \cdot \boldsymbol{i} + v_y \cdot \boldsymbol{j} + v_z \cdot \boldsymbol{k}$:

$$\text{rot}\ \boldsymbol{v} = \nabla \times \boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) \cdot \boldsymbol{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \cdot \boldsymbol{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \cdot \boldsymbol{k} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{bmatrix}$$

Note: The rotation at a point is proportional to the on-axis torque a tiny pinwheel would feel if it were centered at that point.
and also ***Laplacian***, the mathematic operator Δ:

$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial x}$$

Second derivatives of vector functions after applying operator nabla:

$$\text{div}(\text{grad f}) = \nabla \cdot (\nabla \text{f})$$
$$\text{rot}(\text{grad f}) = \nabla \times (\nabla \text{f})$$
$$\Delta \text{f} = \nabla^2 \text{f}$$
$$\text{grad}(\text{div}\ \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v})$$
$$\text{div}(\text{rot}\ \boldsymbol{v}) = \nabla \cdot (\nabla \times \boldsymbol{v})$$

$$\mathrm{rot}(\mathrm{rot}\ \boldsymbol{v}) = \nabla\times(\nabla\times\boldsymbol{v})$$

$$\Delta\boldsymbol{v} = \nabla^2\boldsymbol{v}$$

Some properties of vector functions:

$$\mathrm{rot}(\mathrm{grad\ f}) = \nabla\times(\nabla \mathrm{f}) = 0$$

$$\mathrm{div}(\mathrm{rot}\ \boldsymbol{v}) = \nabla\cdot\nabla\times\boldsymbol{v} = 0$$

$$\mathrm{div}(\mathrm{grad\ f}) = \nabla\cdot(\nabla \mathrm{f}) = \nabla^2 \mathrm{f} = \Delta \mathrm{f}$$

$$\nabla\times\nabla\times\boldsymbol{v} = \nabla(\nabla\cdot\boldsymbol{v}) - \nabla^2\boldsymbol{v}$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$$

$$\nabla\cdot\boldsymbol{v} \neq \boldsymbol{v}\cdot\nabla$$

Using operator nabla ∇ will be next calculation as follows (continuation of equation (9)):

$$\nabla\times(\nabla\times\boldsymbol{H}) = -g\cdot m\cdot\frac{\partial\boldsymbol{H}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{H}}{\partial t^2} \tag{10}$$

$$\nabla(\nabla\cdot\boldsymbol{H}) - \nabla^2\boldsymbol{H} = -g\cdot m\cdot\frac{\partial\boldsymbol{H}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{H}}{\partial t^2} \tag{11}$$

$$\mathrm{grad}(\mathrm{div}\ \boldsymbol{H}) - \nabla^2\boldsymbol{H} = -g\cdot m\cdot\frac{\partial\boldsymbol{H}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{H}}{\partial t^2} \tag{12}$$

Utilizing of 4. Maxwell's equation (4) $\mathrm{div}\ \boldsymbol{B} = 0$, therefore also $\mathrm{div}\ \boldsymbol{H} = 0$, we get:

$$\mathrm{grad}(0) - \nabla^2\boldsymbol{H} = -g\cdot m\cdot\frac{\partial\boldsymbol{H}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{H}}{\partial t^2} \tag{13}$$

$$-\nabla^2\boldsymbol{H} = -g\cdot m\cdot\frac{\partial\boldsymbol{H}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{H}}{\partial t^2} \tag{14}$$

$$\nabla^2\boldsymbol{H} - g\cdot m\cdot\frac{\partial\boldsymbol{H}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{H}}{\partial t^2} = 0 \tag{15}$$

what is wave equation for magnetic component of EMF intensity. By similar procedure, i.e. applying another rotation on 2. Maxwell's equation (2), we obtain analogical wave equation for electric component of EMF intensity, hence:

$$\nabla^2\boldsymbol{E} - g\cdot m\cdot\frac{\partial\boldsymbol{E}}{\partial t} - e\cdot m\cdot\frac{\partial^2\boldsymbol{E}}{\partial t^2} = 0 \tag{16}$$

Equations (15) and (16) create system of general equations of EMW propagation. Their universality result from that fact, that they are valid for any electrical surrounding (*conductive* or *non-conductive*) and for any time flow of EMF variables ***E*** and ***H***. Considering our next utilizing, we customize them in time-harmonic form of both components, therefore time vectors ***E*** and ***H*** will be expressed by rotating phasors in complex plane. Since

$$\underline{\boldsymbol{E}} = \boldsymbol{E}_m\cdot e^{j\cdot w\cdot t}; \qquad \frac{\partial\boldsymbol{E}}{\partial t} = j\cdot w\cdot\boldsymbol{E}_m\cdot e^{j\cdot w\cdot t} = j\cdot w\cdot\underline{\boldsymbol{E}};$$

$$\frac{\partial^2\boldsymbol{E}}{\partial t^2} = -w^2\cdot\boldsymbol{E}_m\cdot e^{j\cdot w\cdot t} = -w^2\cdot\underline{\boldsymbol{E}} \tag{17}$$

similarly

$$\underline{\boldsymbol{H}} = \boldsymbol{H}_m\cdot e^{j\cdot w\cdot t}; \qquad \frac{\partial\boldsymbol{H}}{\partial t} = j\cdot w\cdot\boldsymbol{H}_m\cdot e^{j\cdot w\cdot t} = j\cdot w\cdot\underline{\boldsymbol{H}};$$

$$\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = -w^2 \cdot \boldsymbol{H}_m \cdot e^{j \cdot w \cdot t} = -w^2 \cdot \underline{\boldsymbol{H}} \qquad (18)$$

By utilizing of expressions (17) and (18) the equations (15) and (16) will be in more particular content and after modification will be in form:

$$\nabla^2 \underline{\boldsymbol{H}} + \left(w^2 \cdot e \cdot m - j \cdot w \cdot g \cdot m\right) \cdot \underline{\boldsymbol{H}} = \nabla^2 \underline{\boldsymbol{H}} + k^2 \underline{\boldsymbol{H}} = 0 \qquad (19)$$

$$\nabla^2 \underline{\boldsymbol{E}} + \left(w^2 \cdot e \cdot m - j \cdot w \cdot g \cdot m\right) \cdot \underline{\boldsymbol{E}} = \nabla^2 \underline{\boldsymbol{E}} + k^2 \underline{\boldsymbol{E}} = 0 \qquad (20)$$

They are ***wave equations of harmonic EMF***; they express the propagation of magnetic and electric component of the same harmonic EMW in electrically arbitrary surrounding. Electric properties of surrounding and angular speed, that are in the same two-component of both equations, are expressed by ***propagation constant*** of wave, eventually wave number ***k***, i.e.:

$$\boldsymbol{k}^2 = w^2 \cdot e \cdot m - j \cdot w \cdot g \cdot m = -j \cdot w \cdot m \cdot (g + j \cdot w \cdot e) \qquad (21)$$

therefore in complex plane it has real component and imaginary component

$$\boldsymbol{k} = \sqrt{-j \cdot w \cdot m \cdot (g + j \cdot w \cdot e)} = a - j \cdot b \qquad [\mathrm{rad.m^{-1}}] \qquad (22)$$

Both components, real and positive, will be enumerated by substituting (22) to (21). After arrangement we get:

$$a = w \cdot \sqrt{\frac{m \cdot e}{2} \cdot \left[1 + \sqrt{1 + \left(\frac{g}{w \cdot e}\right)^2}\right]} \qquad [\mathrm{rad.m^{-1}}] \qquad (23)$$

$$b = w \cdot \sqrt{\frac{m \cdot e}{2} \cdot \left[-1 + \sqrt{1 + \left(\frac{g}{w \cdot e}\right)^2}\right]} \qquad [\mathrm{rad.m^{-1}}] \qquad (24)$$

Component a is named ***phase constant*** and component b is named ***attenuation constant***.

At last we modify the constants (22) till (24) for electrically specific surrounding and by them also wave equations of harmonic EMF (19) and (20). We utilize well-known relations between physical constants:

$$c_0^2 = \frac{1}{m_0 \cdot e_0}; \qquad v^2 = \frac{1}{m \cdot e} = \frac{c_0^2}{m_r \cdot e_r}; \qquad l = \frac{v}{f} \qquad (25)$$

in sequence for EMW velocity in vacuum, for EMW in surrounding with permeability m and permitivity e and for wave length l.

After modification we get:

- for electrically ***non-conductive*** surrounding, i.e. $g = 0$:

$$k^2 = w^2 \cdot m \cdot e; \qquad k = w \cdot \sqrt{m \cdot e} = w \cdot \sqrt{m_0 \cdot m_r \cdot e_0 \cdot e_r} = \frac{w}{v};$$

$$a = w \cdot \sqrt{\frac{m \cdot e}{2} \cdot (1+1)} = w \cdot \sqrt{m \cdot e} = k = \frac{w}{v}$$

$$b = w \cdot \sqrt{\frac{m \cdot e}{2} \cdot (1-1)} = 0 \qquad (26)$$

Corresponding wave equations for non-conductive surrounding are:

$$\nabla^2 \underline{\boldsymbol{H}} + w^2 \cdot m \cdot e \cdot \underline{\boldsymbol{H}} = \nabla^2 \underline{\boldsymbol{H}} + a^2 \underline{\boldsymbol{H}} = 0$$

$$\nabla^2 \boldsymbol{E} + w^2 \cdot m \cdot e \cdot \underline{\boldsymbol{E}} = \nabla^2 E + a^2 \underline{\boldsymbol{E}} = 0 \qquad (27)$$

From transformed expressions in system (27) results:

In ideal non-conductive surrounding the electromagnetic wave is not attenuated (b = 0), propagation constant $\boldsymbol{k}$ is reduced to phase constant a, therefore it is real number. Propagation velocity and wave length of EMW in non-conductive surrounding are

$$v = \frac{1}{\sqrt{m \cdot e}} = \frac{w}{a}; \qquad l = \frac{v}{f} = \frac{2 \cdot p}{a} \tag{28}$$

therefore they are dependent on frequency of wave source and on physical properties of surrounding (m a e).

- for electrically ***conductive*** surrounding, i.e. $g > 0$, $g >> w \cdot e$

$$k^2 = -j \cdot w \cdot g \cdot m; \qquad k = \sqrt{-j} \cdot \sqrt{w \cdot g \cdot m} = \frac{1-j}{\sqrt{2}} \cdot \sqrt{w \cdot g \cdot m} = \frac{1-j}{\boldsymbol{a}}$$

$$a = w \cdot \sqrt{\frac{m \cdot e}{2} \cdot \frac{g}{w \cdot e}} = \sqrt{\frac{w \cdot g \cdot m}{2}} = \frac{1}{\boldsymbol{a}}$$

$$b = w \cdot \sqrt{\frac{m \cdot e}{2} \cdot \frac{g}{w \cdot e}} = a = \frac{1}{\boldsymbol{a}} \tag{29}$$

Corresponding wave equations for conductive surrounding are:

$$\nabla^2 \underline{\boldsymbol{H}} - j \cdot w \cdot g \cdot m \cdot \underline{\boldsymbol{H}} = 0$$

$$\nabla^2 \underline{\boldsymbol{E}} - j \cdot w \cdot g \cdot m \cdot \underline{\boldsymbol{E}} = 0 \tag{30}$$

In conductive surrounding the propagation constant of wave $\boldsymbol{k}$ is complex number, phase constant a and attenuation constant b are equal. Physically it means, that conductive surrounding always attenuate the electromagnetic wave. Rate of attenuation is

$$\boldsymbol{a} = \sqrt{\frac{2}{w \cdot g \cdot m}} \qquad [\mathrm{m}] \tag{31}$$

which is obtained by modification of constants (29) and which is named ***the equivalent penetration depth of EMW***. Generally it provide the representation of surrounding influence (g and m) and source frequency of EMF to its arrangement in objective conductive surrounding. Propagation velocity and wave length of EMW in this surrounding are also functions of penetration depth, because

$$v = \frac{w}{a} = w \cdot \boldsymbol{a} = 2 \cdot p \cdot f \cdot \boldsymbol{a}; \qquad l = \frac{v}{f} = 2 \cdot p \cdot \boldsymbol{a} \tag{32}$$

When is smaller penetration depth of EMW into conductive surrounding, then its velocity and wave length is smaller. Physically, the penetration depth corresponds to distance from body surface, in which the plane electromagnetic wave is attenuated in 95 % of intensity on surface.

Example 1

Determine the penetration depth of electromagnetic wave into copper at frequencies 50 Hz, 500 Hz, 10 kHz, 100 kHz, 1 MHz a 10 MHz. Electric conductivity of copper is $6{,}43.10^7$ S/m, relative permeability is equals to 1. What is the resistance of circular solid conductor with diameter 2 mm at these frequencies in comparison to its resistance flowed by direct current?

Solution:

For penetration depth is valid expression:

$$\boldsymbol{a} = \sqrt{\frac{2}{w \cdot g \cdot m}} = \sqrt{\frac{2}{2 \cdot p \cdot f \cdot g \cdot m_0 \cdot m_r}} = \sqrt{\frac{1}{p \cdot f \cdot g \cdot 4 \cdot p \cdot 10^{-7} \cdot m_r}} = \frac{1}{2 \cdot p} \cdot \frac{1}{\sqrt{g \cdot 10^{-7}}} \cdot \frac{1}{\sqrt{f}} \cong \frac{0{,}0628}{\sqrt{f}}$$

After substituting particular frequencies we get these results:

Frequency [kHz]	0,05	0,5	10	100	1000	10000
Penetration depth [mm]	8,87	2,81	0,628	0,198	0,0628	0,0198

For comparison of resistance in specific and zero frequency we start from the physical meaning of penetration depth for plane conductor. In consequence of skin-effect the conductor behaves quasi the current flows by uniformly distributed current density in the layer of penetration depth $\boldsymbol{a}$. We can it apply on cylindrical conductor only in case, that penetration depth is significantly smaller as its radius. According to previous table it can be approximately satisfied at frequency 100 kHz. For higher frequencies it is valid more precisely, if there is higher frequency. Proportion of resistances will be equal to reciprocal proportion of cross-section areas, in that is flowed uniformly distributed current

$$\frac{R}{R_0} = \frac{S_0}{S} \cong \frac{p \cdot r^2}{2 \cdot p \cdot r \cdot \boldsymbol{a}} = \frac{r}{2 \cdot \boldsymbol{a}}$$

Values of this proportion introduces next table, for frequency 100 kHz is necessary this value regard only as rough calculation, because condition $\boldsymbol{r} >> \boldsymbol{a}$ is not satisfied, $\boldsymbol{r} = 5 \cdot \boldsymbol{a}$.

Frequency [kHz]	100	1000	10000
$\boldsymbol{R/R_0}$	2,5	8	25

Previous formula for resistance proportion calculates area of ring approximately so, that it is spread in form of strip and it supposes, that it creates rectangular. In fact, it is rhomboid. Exact area of ring is defined in following manner:

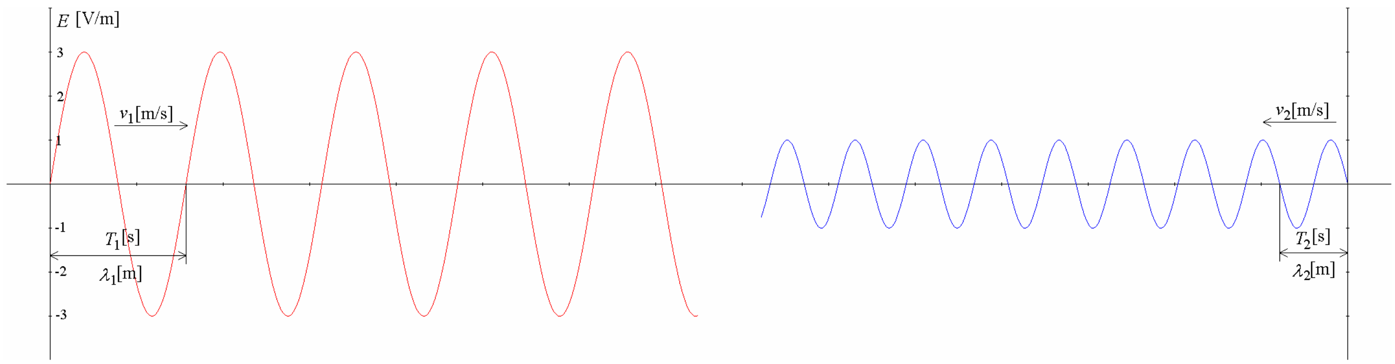
$$S = p \cdot \left[r^2 - (r - \boldsymbol{a})^2\right] = p \cdot \left[r^2 - \left(r^2 - 2 \cdot r \cdot \boldsymbol{a} + \boldsymbol{a}^2\right)\right] = p \cdot \left(r^2 - r^2 + 2 \cdot r \cdot \boldsymbol{a} - \boldsymbol{a}^2\right) = 2 \cdot p \cdot r \cdot \boldsymbol{a} - p \cdot \boldsymbol{a}^2 \cong$$
$$\cong 2 \cdot p \cdot r \cdot \boldsymbol{a}$$

Last expression is valid for $r \cdot \boldsymbol{a} << \boldsymbol{a}^2$, i.e. $r << \boldsymbol{a}$. Calculation with exact cross-section area is naturally useless, because for radius, that is comparable to penetration depth, there is not satisfied initial condition, that current is flowed with uniformly distributed current density only in layer of penetration depth $\boldsymbol{a}$.

Example 2

Two transmitting antennas in remote distance 50 km transmit in the same time signal. Surrounding between particular transmitters is air. Frequency of EM radiation of the first transmitter is 900 MHz, the second one 1800 MHz. Radiant magnitude value of electrical component of the first transmitter is 3 V/m, the second one 1 V/m. Determine the propagation constants, phase constants, attenuation constants, wave lengths, propagation velocities, penetration depths of particular EM waves and immediate value of electric field intensity in time of wave interference, immediate value of electric field intensity in time $t = 10^{-6}$ s, number of periods of particular waves in time of wave interference.

Solution:



Propagation constant:

$$k_1 = w_1 \cdot \sqrt{m \cdot e} = 2 \cdot p \cdot f_1 \cdot \sqrt{m_0 \cdot m_{\mathrm{r}} \cdot e_0 \cdot e_{\mathrm{r}}}$$

$$k_1 = 2 \cdot p \cdot 900 \cdot 10^6 \cdot \sqrt{4 \cdot p \cdot 10^{-7} \cdot 1 \cdot 8{,}854 \cdot 10^{-12} \cdot 1} = 18{,}862\ \mathrm{rad \cdot m^{-1}}$$

$$k_2 = w_2 \cdot \sqrt{m \cdot e} = 2 \cdot p \cdot f_2 \cdot \sqrt{m_0 \cdot m_{\mathrm{r}} \cdot e_0 \cdot e_{\mathrm{r}}}$$

$$k_2 = 2 \cdot p \cdot 1800 \cdot 10^6 \cdot \sqrt{4 \cdot p \cdot 10^{-7} \cdot 1 \cdot 8{,}854 \cdot 10^{-12} \cdot 1} = 37{,}725\ \mathrm{rad \cdot m^{-1}}$$

Phase constant and *attenuation constant*:

As there deals about non-conductive surrounding, in which surrounding conductivity $g = 0$, then:

$$a_1 = k_1 = 18{,}862\ \mathrm{rad \cdot m^{-1}}; \qquad b_1 = 0$$

$$a_2 = k_2 = 37{,}725\ \mathrm{rad \cdot m^{-1}}; \qquad b_2 = 0$$

Length of period of particular waves:

$$T_1 = \frac{1}{f_1} = \frac{1}{900 \cdot 10^6} = 1{,}111 \cdot 10^{-9}\ \mathrm{s}$$

$$T_2 = \frac{1}{f_2} = \frac{1}{1800 \cdot 10^6} = 5{,}555 \cdot 10^{-10}\ \mathrm{s}$$

Propagation velocity of particular waves:

$$v_1 = \frac{1}{\sqrt{m \cdot e}} = \frac{w_1}{a_1} = \frac{1}{\sqrt{m_0 \cdot m_{\mathrm{r}} \cdot e_0 \cdot e}} = \frac{1}{\sqrt{4 \cdot p \cdot 10^{-7} \cdot 1 \cdot 8{,}854 \cdot 10^{-12} \cdot 1}} = c_0 = 2{,}998 \cdot 10^8\ \mathrm{m \cdot s^{-1}}$$

$$v_2 = \frac{1}{\sqrt{m \cdot e}} = \frac{w_2}{a_2} = \frac{1}{\sqrt{m_0 \cdot m_{\mathrm{r}} \cdot e_0 \cdot e}} = \frac{1}{\sqrt{4 \cdot p \cdot 10^{-7} \cdot 1 \cdot 8{,}854 \cdot 10^{-12} \cdot 1}} = c_0 = 2{,}998 \cdot 10^8\ \mathrm{m \cdot s^{-1}}$$

Wave length:

$$l_1 = v_1 \cdot T_1 = \frac{v_1}{f_1} = \frac{w_1}{a_1 \cdot f_1} = \frac{2 \cdot p}{a_1} = 2{,}998 \cdot 10^8 \cdot 1{,}111 \cdot 10^{-9} = 0{,}333\ \text{m}$$

$$l_2 = v_2 \cdot T_2 = \frac{v_2}{f_2} = \frac{w_2}{a_2 \cdot f_2} = \frac{2 \cdot p}{a_2} = 2{,}998 \cdot 10^8 \cdot 5{,}555 \cdot 10^{-10} = 0{,}167\ \text{m}$$

Penetration depth of particular waves at specific frequencies:
As attenuation constant of both waves in non-conductive surrounding is equals to zero, wave is not attenuated:

$$a_1 = \sqrt{\frac{2}{w_1 \cdot g \cdot m}} = \sqrt{\frac{2}{2 \cdot p \cdot 900 \cdot 10^6 \cdot 0 \cdot 4 \cdot p \cdot 10^{-7} \cdot 1}} = \lim \sqrt{\frac{2}{0}} = \infty\ \text{m}$$

$$a_2 = \sqrt{\frac{2}{w_2 \cdot g \cdot m}} = \sqrt{\frac{2}{2 \cdot p \cdot 1800 \cdot 10^6 \cdot 0 \cdot 4 \cdot p \cdot 10^{-7} \cdot 1}} = \lim \sqrt{\frac{2}{0}} = \infty\ \text{m}$$

Time, when there happens the wave interference:

$$t_{\text{int}} = t_1 + t_2 = \frac{s}{v_1 + v_2} = \frac{50 \cdot 10^3}{2{,}998 \cdot 10^8 + 2{,}998 \cdot 10^8} = 8{,}339 \cdot 10^{-5}\ \text{s}$$

Immediate value of electric field intensity in time of wave interference (precision of these results is in considerable rate dependent on previous rounding):

$$E_1(t_{\text{int}}) = E_{1\max} \cdot \sin(w_1 \cdot t_{\text{int}}) = E_{1\max} \cdot \sin(2 \cdot p \cdot f_1 \cdot t_{\text{int}}) = 3 \cdot \sin(2 \cdot p \cdot 900 \cdot 10^6 \cdot 8{,}339 \cdot 10^{-5}) = -1{,}955\ \text{V} \cdot \text{m}^{-1}$$

$$E_2(t_{\text{int}}) = E_{2\max} \cdot \sin(w_2 \cdot t_{\text{int}}) = E_{2\max} \cdot \sin(2 \cdot p \cdot f_2 \cdot t_{\text{int}}) = 1 \cdot \sin(2 \cdot p \cdot 1800 \cdot 10^6 \cdot 8{,}339 \cdot 10^{-5}) = -0{,}989\ \text{V} \cdot \text{m}^{-1}$$

Immediate value of electric field intensity in time of $t_{\text{v}} = 10^{-6}$ s:

$$E_1(t_{\text{v}}) = E_{1\max} \cdot \sin(2 \cdot p \cdot f_1 \cdot t_{\text{v}}) = 3 \cdot \sin(2 \cdot p \cdot 900 \cdot 10^6 \cdot 10^{-6}) = -2{,}896\ \text{V} \cdot \text{m}^{-1}$$

$$E_2(t_{\text{v}}) = E_{2\max} \cdot \sin(2 \cdot p \cdot f_2 \cdot t_{\text{v}}) = 1 \cdot \sin(2 \cdot p \cdot 1800 \cdot 10^6 \cdot 10^{-6}) = 0{,}504\ \text{V} \cdot \text{m}^{-1}$$

Number of periods of particular waves in time of wave interference:

$$n_1 = \frac{t_{\text{int}}}{T_1} = \frac{8{,}339 \cdot 10^{-5}}{1{,}111 \cdot 10^{-9}} = 75059 \qquad \text{or} \qquad n_1 = \frac{s_1}{l_1} = \frac{25000}{0{,}333} = 75075\ \text{(rounding error)}$$

$$n_2 = \frac{t_{\text{int}}}{T_2} = \frac{8{,}339 \cdot 10^{-5}}{5{,}555 \cdot 10^{-10}} = 150117 \qquad \text{or} \qquad n_1 = \frac{s_1}{l_1} = \frac{25000}{0{,}167} = 149701\ \text{(rounding error)}$$